M303/D

M3031606F1

Module Examination 2016
Further Pure Mathematics

Wednesday 1 June 2016 **2.30 pm – 5.30 pm**

Time allowed: 3 hours

Instructions

The exam is in two parts. Attempt *all* questions in Part 1 (each question is worth 11 marks) and *two* out of the three questions in Part 2 (each question is worth 17 marks). If you submit attempts to all three questions in Part 2, your best two scores will count towards your result.

It should take about two hours to do the Part 1 questions and an hour to do two Part 2 questions.

Start each question on a new page.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles $\circ$ to indicate any 'corner' points on a boundary that are not included and small closed circles $\bullet$ to indicate any 'corner' points that are included.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is *not* permitted in this examination.

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PART 1

Attempt all questions. Each question is worth 11 marks. All of the Part 1 questions count towards your result.

Question 1

(a) Prove by induction that

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

for all positive integers n. [4]

(b) (i) Use the Euclidean algorithm to find $\text{hcf}(231, 715)$. [3]

(ii) Hence find integers s and t such that

$$231s + 715t = \text{hcf}(231, 715).$$ [4]

Question 2

(a) Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$.

(i) Write down the order of G. [1]

(ii) Write down the maximum order of an element in G. [1]

(iii) Find the number of elements of order 3 in G. [2]

(b) Let G be a group with proper subgroups H and K.

Determine whether each of the following statements is true or false, and either give a proof or a counter-example.

(i) If H and K are normal in G, then $H \cap K$ is normal in G.
You may assume that $H \cap K$ is a subgroup of G.

(ii) If G is abelian, then H is cyclic. [4]

(c) Let ϕ be the homomorphism

$$\begin{aligned} \phi\colon \mathbb{Z}_6 &\to \mathbb{Z}_6 \\ x &\mapsto x + x \end{aligned}$$

Find $\text{Im}(\phi)$ and $\text{Ker}(\phi)$.

You are NOT asked to prove that ϕ is a homomorphism. [3]

Question 3

(a) Calculate the following Legendre symbols (note that 101 and 103 are prime)

(i) $(3/103)$,

(ii) $(101/103)$,

(iii) $(303/103)$. [5]

(b) Use the rational root test to show that

$$f(x) = x^3 + 17x^2 - 51x + 3$$

is irreducible over $\mathbb{Q}$. [5]

(c) Can 51 be expressed as the sum of two squares? Briefly justify your answer. [1]

Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 4\}$$
$$B = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 > 1\}$$
$$C = A \cap B.$$

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Write down in set notation the closure, interior and boundary of C for the following metrics:

(i) the Euclidean metric $d^{(2)}$ for the plane; [3]

(ii) the discrete metric d_0 for the plane. [3]

(c) Show that C is not compact for the Euclidean metric. [2]

Question 5

Let $\alpha = \sqrt{5} - \sqrt{2}$; you may assume that
$[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$, $\sqrt{2} \notin \mathbb{Q}(\sqrt{5})$ and $\sqrt{5} \notin \mathbb{Q}(\sqrt{2})$.

(a) Express α^3 in the form $a\sqrt{2} + b\sqrt{5}$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of part (a) or otherwise, show that both $\sqrt{2}$ and $\sqrt{5}$ are in $\mathbb{Q}(\alpha)$. [3]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. [1]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [3]

Question 6

Define a metric, d, on $X = \mathbb{R}^2 - \{(x, y) : y = 1\}$ by

$$d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.

You do not have to show that d is a metric.

(a) Write down $d\big((3, 7), (1, 2)\big)$ [*1*]

(b) Write down a d-disconnection of X. [*2*]

(c) Show that the sequence $\mathbf{a}_n = \left(\dfrac{\cos n}{n}, \dfrac{\sin n}{n}\right)$ is a d-convergent sequence. [*3*]

(d) Show that $\mathbb{R}^2 - \{(x, y) : y = 1\}$ is not d-complete. [*4*]

(e) Deduce that X is not d-sequentially compact. [*1*]

PART 2

Attempt two questions. Each question is worth 17 marks. (If you submit attempts to all three questions, your best two scores will count towards your result.)

Question 7

Let G be a group of order 3^2q, where q is prime and $q > 3$.

(a) Show that G has a unique Sylow q-subgroup. [3]

(b) Let $q = 11$.

(i) Find the possible number of Sylow 3-subgroups of G. [2]

(ii) Use parts (a) and (b)(i) to deduce that G is abelian. [5]

(iii) Hence classify all groups of order 99. [3]

(c) Does the argument in part (b) generalise to show that all groups of order 3^2q, where q is prime and $q > 3$, are necessarily abelian? Justify your answer. [3]

(d) Give an example of a non-abelian group of order p^2r, where p and r are prime and $p < r$. [1]

Question 8

Let $N(a + bi) = a^2 + b^2$ be the usual Euclidean norm on the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$.

(a) Show that the only elements $r \in \mathbb{Z}[i]$ for which $N(r) = 2$ are $1 + i$, $1 - i$, $-1 + i$ and $-1 - i$. [3]

(b) Let r be a common factor of $s, t \in \mathbb{Z}[i]$. Show that $N(r)$ is a common factor of $N(s)$ and $N(t)$ in $\mathbb{Z}[i]$. [2]

(c) Hence show that $1 + i$ is an hcf of $3 + i$ and 6 in $\mathbb{Z}[i]$. Briefly explain why $1 - i$, $-1 + i$ and $-1 - i$ are also hcfs of $3 + i$ and 6.

Let $I = \langle 6, 3 + i \rangle$ be the ideal in $\mathbb{Z}[i]$ generated by the elements 6 and $3 + i$. Since $\mathbb{Z}[i]$ is a principal ideal domain, the ideal I is principal. [4]

(d) Show that $\{1 + i, 1 - i, -1 + i, -1 - i\}$ is the set of all elements $d \in \mathbb{Z}[i]$ such that $I = \langle d \rangle$. [3]

(e) Using the results of parts (c) and (d) or otherwise, show that $I = \{\alpha + \beta i;\ \alpha, \beta \in \mathbb{Z},\ \alpha \equiv \beta \pmod 2\}$. [3]

(f) With the help of the previous parts or otherwise, show that I is a maximal ideal in $\mathbb{Z}[i]$. [2]

Question 9

This question concerns the set $C[0,1]$ with the max metric $d_{\max}$.

(a) Let (f_n) be the sequence of functions where $f_n\colon [0,1] \to \mathbb{R}$ is given by $f_n(x) = \frac{1}{n}\big(|\sin(\frac{\pi}{2}x)| + |\cos(\frac{\pi}{2}x)|\big)$.

(i) Write down $f_n(0)$ and $f_n(1)$. [2]

(ii) Prove that (f_n) is $d_{\max}$-convergent to f_0, where f_0 denotes the zero function. [3]

Now consider the set
$S = \{f \in C[0,1] : f(x) > 0 \text{ for each } x \in [0,1]\}$

(iii) Show that $f_n \in S$ for each $n \in \mathbb{N}$. [3]

(iv) Hence, or otherwise, show that S is not $d_{\max}$-compact. [2]

(b) Let $A = \{f \in C[0,1] : |f(x) - f(y)| \leq 2|x-y| \text{ for each } x, y \in [0,1]\}$ and $B = \{f \in C[0,1] : |f(x)| \leq 2 \text{ for each } x \in [0,1]\}$.

(i) Show that A is $d_{\max}$-closed. [3]

(ii) Deduce that $A \cap B$ is $d_{\max}$-closed. [2]

(iii) Hence, or otherwise, show that $A \cap B$ is $d_{\max}$-sequentially compact. [2]

[END OF QUESTION PAPER]